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A NUMERICAL STUDY OF THE STEADY SCALAR  
CONVECTIVE DIFFUSION EQUATION FOR SMALL  
VISCOSITY

Michael B. Giles  
and  
Milton E. Rose

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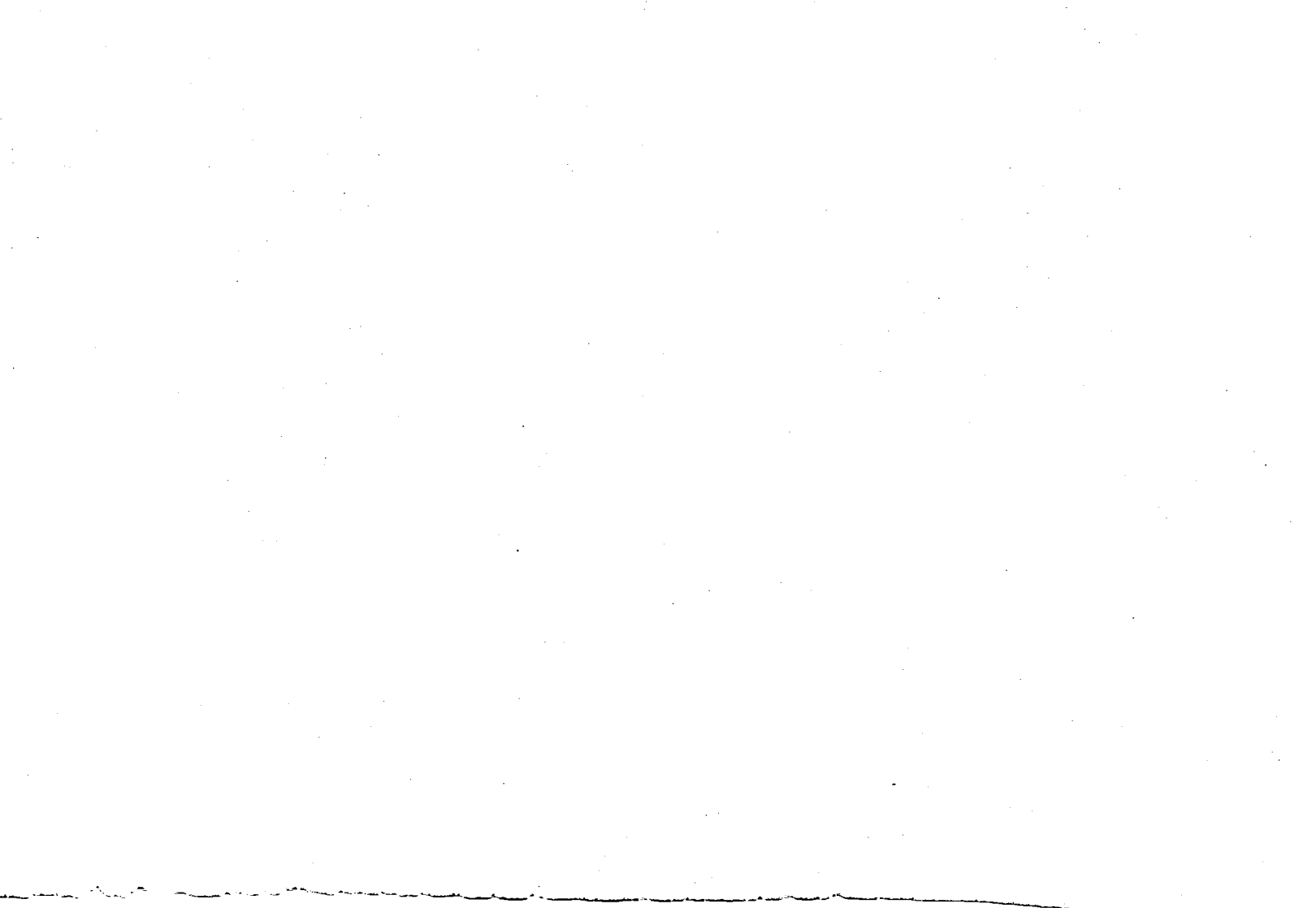
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MINS: / BOUNDARY LAYERS/ FINITE DIFFERENCE THEORY/ INVISCID FLOW/ NAVIER-STOKES EQUATION/ NUMERICAL ANALYSIS/ PERTURBATION

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ABS: A time-independent convection diffusion equation is studied by means of a compact finite difference scheme and numerical solutions are compared to the analytic inviscid solutions. The correct internal and external boundary layer behavior is observed, due to an inherent feature of the scheme which automatically produces upwind differencing in inviscid regions and the correct viscous behavior in viscous regions.

ENTER:



A NUMERICAL STUDY OF THE STEADY SCALAR  
CONVECTIVE DIFFUSION EQUATION FOR SMALL VISCOSITY

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Abstract

The equation  $\nu \nabla^2 u = f_x(u) + g_y(u)$  is studied by means of a compact finite difference scheme and numerical solutions are compared to the analytic inviscid ( $\nu = 0$ ) solutions. The correct internal and external boundary layer behaviour is observed, due to an inherent feature of the scheme which automatically produces upwind differencing in inviscid regions and the correct viscous behaviour in viscous regions.

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## INTRODUCTION

Consider a domain  $D$  with boundary  $\Gamma$  in the  $x y$  plane, in which  $u(v)$  is the solution of the time-independent convection diffusion equation

$$f_x(u) + g_y(u) = v \nabla^2 u \quad (1.1)$$

for  $u = \bar{u}$  on  $\Gamma$ . Certain features of this problem can provide useful insights into the Navier-Stokes equations and their inviscid limit, the Euler equations (we employ terminology suggested by these applications in the following discussion). In fluid dynamics it is common to obtain the time-independent solutions to these problems by solving the time dependent problem for large times. In this paper we instead study (1.1) by a time-independent finite difference equation to which rather standard iterative methods will be shown to apply.

A large variety of finite difference methods can be employed to solve this problem in the sense that if  $u(v, h)$  indicates the solution of a finite difference scheme in which  $h$  is a typical mesh length then  $u(v, h) \rightarrow u(v)$  as  $h \rightarrow 0$  in, say, an  $L_2$  norm. In particular, integrating (1.1) over a subdomain  $D'$  with boundary  $\Gamma'$  produces the conservative integral formulation

$$\oint_{\Gamma'} (f n_x + g n_y) ds = \oint_{\Gamma'} v \nabla u \cdot \underline{n} ds, \quad (1.2)$$

where  $\underline{n} = (n_x, n_y)^T$  is the unit normal vector. Conservative finite difference schemes approximate this equation in each cell. Nonconservative schemes instead approximate the equivalent differential equation

$$a u_x + b u_y = v \nabla^2 u, \quad (1.3)$$

where  $a = \partial f / \partial u$  and  $b = \partial g / \partial u$  in each cell. Since  $u(v)$  is smooth both approaches produce the correct solution in the limit  $h \rightarrow 0$ .

The situation is more complicated for the "inviscid" weak solution  $u^0$  which satisfies

$$\oint_{\Gamma'} (f n_x + g n_y) ds = 0, \quad (1.4)$$

for all closed curves  $\Gamma'$ . This is equivalent to the hyperbolic problem

$$a u_x^0 + b u_y^0 = 0, \quad (1.5)$$

provided  $u^0$  is differentiable. However, in the inviscid case boundary data may only be prescribed on part of  $\Gamma$ , say  $\Gamma^0$ . Furthermore there may be nonunique solutions with discontinuities (shocks) or closed characteristic curves (recirculating flows) if  $a^2 + b^2 = 0$  at points in  $D$ .

Many of the essential facts concerning the relationship of the solution  $u(v)$  of (1.1) with the "physically relevant" solution  $u^0$  of (1.4) are suggested by formal singular perturbation arguments. Specifically, in the limit  $v \rightarrow 0$ , we expect that the solution  $u(v)$  of (1.1) converges to the "physical" weak solution  $u^0$  of (1.4) and boundary layers arise on the complement of  $\Gamma^0$  in  $\Gamma$ , i.e., that part of  $\Gamma$  on which no boundary data is required for the inviscid problem for  $u^0$ .

The use of finite difference schemes to calculate the solution  $u(v)$  in the limit  $v \rightarrow 0$  introduces a second limiting process  $h \rightarrow 0$ . It is natural to consider the different limiting paths by which  $u(v, h) \rightarrow u^0$  as suggested by the following figure:



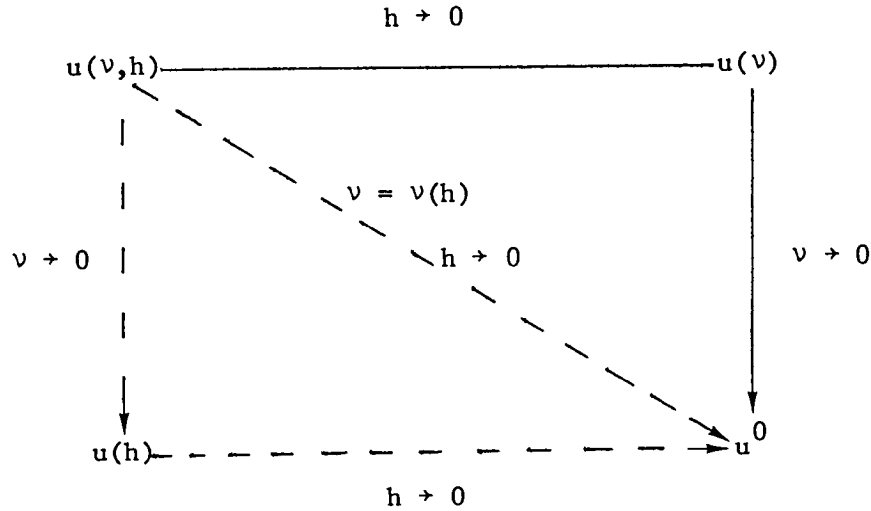


Figure 1: Typical limiting processes considered in this paper.

The solid arrows indicate limiting processes which are known to be convergent, assuming a consistent stable finite difference scheme. The dotted arrows denote limiting processes which in general may not be convergent, or may converge to a solution different from  $u^0$ .

This paper is concerned with investigating a finite difference scheme which offers useful insights into this problem. The scheme has its origin in a compact exponential scheme described by Phillips and Rose [1] for time-dependent problems; an application of this scheme to the Navier-Stokes equations was also described by Rose [3]. An effective time-independent solution method is provided by employing the flux-elimination technique described in Phillips and Rose [2]. The first few sections describe the application of these developments to equation (1.1). The relationship of the solution  $u(v, h)$  to  $u^0$  is then explored by several numerical experiments and simple heuristic analyses.

## 2. A COMPACT SCHEME

Consider (1.1) in system form

$$v_x + w_y = f_x(u) + g_y(u)$$

$$v = v u_x \quad (2.1)$$

$$w = v u_y.$$

In a square cell  $\pi_{i,j}$ , with sides of length  $2h$ ,  $u_{i\pm 1/2,j}$ ,  $u_{i,j\pm 1/2}$ ,  $v_{i\pm 1/2,j}$ ,  $w_{i,j\pm 1/2}$  represent the average values of the analytic variables on the corresponding sides of  $\pi_{i,j}$ .

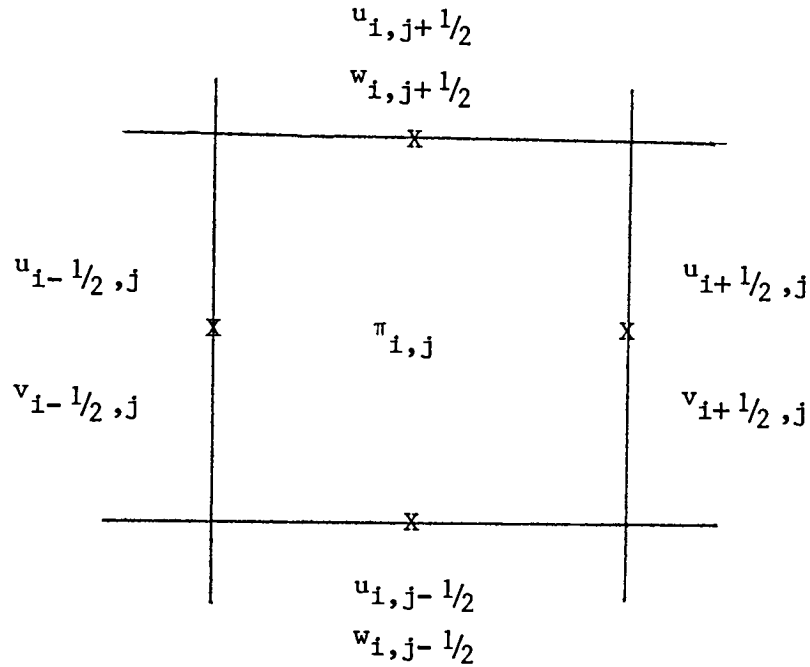


Figure 2: Location of finite difference variables in a cell  $\pi_{i,j}$ .

Because solutions of (2.1) are smooth, if  $h$  is sufficiently small any solution of (2.1) can be approximated locally by

$$v \nabla^2 u = a u_x + b u_y$$

where  $a, b$  indicate average values of  $f_u, g_u$  in the cell. This equation has elementary solutions  $u = \exp(\alpha x + \beta y)$  where  $\alpha(a - v\alpha) + \beta(b - v\beta) = 0$ . Any linear combination of four such solutions can be used to obtain a solution having the prescribed average values  $u_{i\pm 1/2, j}, u_{i, j\pm 1/2}$  on the sides of  $\pi_{i, j}$ ; the corresponding average values  $v_{i\pm 1/2, j}, w_{i, j\pm 1/2}$  are then related by four algebraic equations. The following algebraic system arises when a solution of the form

$$u = c_1 + c_2 \exp\left(\frac{ax}{v}\right) + c_3 \exp\left(\frac{by}{v}\right) + c_4 \exp\left(\frac{ax + by}{v}\right)$$

is considered.

$$a) \quad \delta_x v + \delta_y w = \delta_x f + \delta_y g$$

$$b) \quad (\mu_x - h p_x \delta_x) v = v \delta_x u$$

(2.2)

$$c) \quad (\mu_y - h p_y \delta_y) w = v \delta_y u$$

$$d) \quad h^2 (q_x \delta_x v - q_y \delta_y w) = v (\mu_x - \mu_y) u$$

where

$$\delta_x u = (u_{i+1/2, j} - u_{i-1/2, j}) / 2h$$

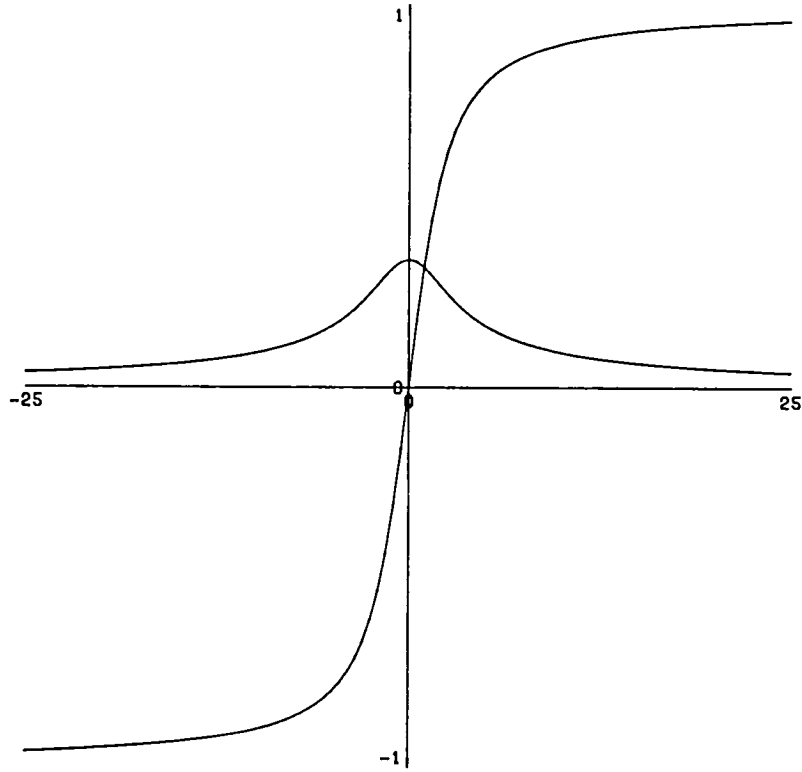
$$\mu_x u = (u_{i+1/2, j} + u_{i-1/2, j}) / 2,$$

etc., and, if  $\rho = h/v$ ,  $\theta_x = a\rho$ ,  $\theta_y = b\rho$ , then

$$p(\theta) = \coth \theta - \theta^{-1} \quad p_x \equiv p(\theta_x), \quad p_y \equiv p(\theta_y) \quad (2.3)$$

$$q(\theta) = \theta^{-1} p(\theta) \quad q_x \equiv q(\theta_x), \quad q_y \equiv q(\theta_y). \quad (2.4)$$

The simple behaviour of  $p(\theta)$ ,  $q(\theta)$  are shown in Figure 3.



**Figure 3.** Graph of  $p(\theta)$ ,  $q(\theta)$ .

The dimensionless parameters  $\theta_x$ ,  $\theta_y$  are the cell Reynolds numbers and provide a measure of the relative importance of diffusion versus transport effects in a cell. An important feature of this scheme is the manner in which

upwind differencing arises when  $|\theta| \rightarrow \infty$ . From (2.3), (2.4),

$$\begin{aligned}
 p(\theta) &\sim \operatorname{sgn}(\theta) & |\theta| &\text{ large} \\
 &\sim \theta/3 & |\theta| &\text{ small}
 \end{aligned}
 \tag{2.5}$$

$$\begin{aligned}
 q(\theta) &\sim |\theta|^{-1} & |\theta| &\text{ large} \\
 &\sim 1/3 & |\theta| &\text{ small.}
 \end{aligned}$$

In (2.2b)  $(\mu_x - h p_x \delta_x) v$  represents a weighted average of the values  $v_{i \pm 1/2, j}$  on the two cell sides. For  $|\theta| \rightarrow \infty$  the weights tend to 0 or 1 thereby selecting one or another of the values depending on the sign of  $\theta$ . For  $|\theta| \rightarrow 0$  the weights tend to  $1/2$  giving a simple central average  $\mu_x v$ .

An important point to note is that equation (2.2a) can be interpreted as a finite difference form of the conservative integral equation (1.2). Summation over all the cells in domain  $D$  yields

$$\sum_{\Gamma} (f n_x + g n_y) h = \sum_{\Gamma} (v n_x + w n_y) h, \tag{2.6}$$

so the scheme is globally conservative. This is important in ensuring the correct calculation of nonlinear shocks, as will be shown later.

The scheme (2.2) is called compact because it involves only the values of  $u, v, w$  associated with an individual cell. Any relationships between values in other cells arise from imposing the further condition that values are continuous across cell boundaries. When  $a$  and  $b$  are constant in all cells an energy estimate similar to that in [1], [2] can be obtained from (2.2); the result is that the solution  $u(v, h)$  converges as  $h \rightarrow 0$  to a smooth solution  $u(v)$  of (2.1) with  $O(h^2)$  accuracy.

### 3. FLUX ELIMINATION

Let

$$\begin{aligned} V &= (\mu_x v, \Delta_x v, \mu_y w, \Delta_y w)^T \\ U &= (\mu_x u, \Delta_x u, \mu_y u, \Delta_y u)^T \end{aligned} \quad (3.1)$$

where  $\Delta = h \delta$ . Equations (2.2) may be solved for  $V$  to obtain

$$V = R(U) \quad (3.2)$$

where

$$R(U) = \sigma^{-1} \begin{bmatrix} \sigma \Delta_x u + p_x [(\mu_x - \mu_y)u + \rho q_y (\Delta_x f + \Delta_y g)] \\ (\mu_x - \mu_y)u + \rho q_y (\Delta_x f + \Delta_y g) \\ \sigma \Delta_y u + p_y [(\mu_y - \mu_x)u + \rho q_x (\Delta_x f + \Delta_y g)] \\ (\mu_y - \mu_x)u + \rho q_x (\Delta_x f + \Delta_y g) \end{bmatrix} \quad (3.3)$$

in which  $\sigma = q_x + q_y$ .

Following the method outlined in Phillips and Rose [2] consider two neighboring cells  $\pi_{ij}$  and  $\pi_{i+1,j}$  having the common values  $u_{i+1/2,j}$  and  $v_{i+1/2,j}$  associated with their common side. Clearly

$$v_{i+1/2,j} = (\mu_x + \Delta_x) v_{i,j} = (\mu_x - \Delta_x) v_{i+1,j}$$

so that

$$\begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} R_{i,j}(U) = \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix} R_{i+1,j}(U).$$

Similarly considering the value  $w_{i,j+1/2}$  common to cells  $\pi_{i,j}$  and  $\pi_{i,j+1}$  we also have

$$\begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} R_{i,j}(U) = \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix} R_{i,j+1}(U).$$

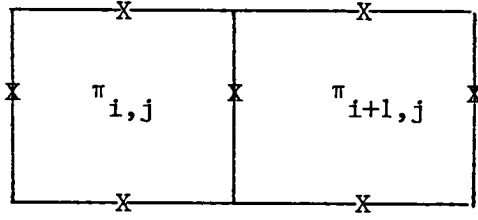
Using (3.3) there results

$$\begin{aligned} \text{a) } \mu_x \{ \sigma^{-1} [ (\mu_x - \mu_y)u + \rho q_y (\Delta_x f + \Delta_y g) ] \} \\ - \Delta_x \{ \Delta_x u + p_x \sigma^{-1} [ (\mu_x - \mu_y)u + \rho q_y (\Delta_x f + \Delta_y g) ] \} = 0 \end{aligned} \quad (3.4)$$

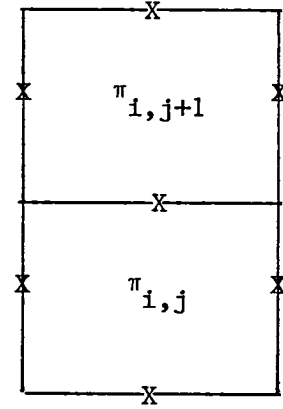
$$\begin{aligned} \text{b) } \mu_y \{ \sigma^{-1} [ (\mu_y - \mu_x)u + \rho q_x (\Delta_x f + \Delta_y g) ] \} \\ - \Delta_y \{ \Delta_y u + p_y \sigma^{-1} [ (\mu_y - \mu_x)u + \rho q_x (\Delta_x f + \Delta_y g) ] \} = 0. \end{aligned}$$

The values of  $u$  related by each of these equations are indicated in Figure 4. We refer to (3.4) as the flux-eliminated form of (2.2). From its solution  $v$  and  $w$  may be calculated from (3.2).

When only  $u$  is prescribed on the boundary then (3.4) applies as indicated at all interior points. When, more generally,  $v$  or  $w$  are prescribed at boundary points additional equations expressing these prescribed values are obtained from (3.2). Finally we call attention to the fact that the parameters  $h$  and  $v$  occur in (3.4) only in the combination  $\rho = h/v$ .



(a)



(b)

Figure 4: Stencils for (3.4a,b)

#### 4. LIMITING FORMS OF FLUX-ELIMINATED EQUATIONS

As stated in the introduction we shall be interested in studying the solution  $u(v,h)$  of (3.4) as  $h \rightarrow 0$ ,  $v(h) \rightarrow 0$ . Depending on the function  $v(h)$ , as  $h \rightarrow 0$ ,  $\rho = h/v$  approaches 0, a constant, or  $\infty$ . In this section we shall describe some results of formally applying the limits  $\rho \rightarrow 0, \infty$  in (3.4) using the asymptotic values for  $p, q$  given in (2.5). For our purpose it will be sufficient to examine only (3.4a).

Case 1:  $\rho \rightarrow 0$

When  $\theta \simeq 0$  then  $p \simeq 0$ ,  $q \simeq 1/3$  and (3.4a) reduces to

$$(u_{i-1/2,j} + 10 u_{i+1/2,j} + u_{i+3/2,j})$$

$$- 3(u_{i,j-1/2} + u_{i,j+1/2} + u_{i+1,j-1/2} + u_{i+1,j+1/2}) \simeq 0.$$



Case 2:  $\rho \rightarrow \infty$

When  $|\theta| \gg 1$  then  $p \sim \text{sgn}(\theta)$ ,  $q \sim |\theta|^{-1}$  and so in this case different limiting forms arise depending on  $\text{sgn}(a)$ ,  $\text{sgn}(b)$  where  $a = f_u$ ,  $b = g_u$ . We write

$$a_- \equiv a_{i,j} \qquad a_+ \equiv a_{i+1,j}$$

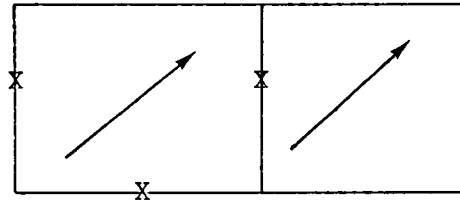
$$b_- \equiv b_{i,j} \qquad b_+ \equiv b_{i+1,j}$$

$$(i) \quad a_{\pm} > 0, \quad b_{\pm} > 0$$

Equation (3.4a) reduces to

$$(b_- - a_-)u_{i-1/2,j} + (b_- + a_-)u_{i+1/2,j} - 2b_- u_{i,j-1/2} \simeq 0.$$

The relevant stencil is shown below with arrows indicating the direction of the inviscid characteristic. Note that the reduced equation uses only "upwind" information to calculate  $u_{i+1/2,j}$ . In particular if these two cells are the entire computational domain then the prescribed boundary values on the upwind side of  $u_{i+1/2,j}$  solely determine its values. This corresponds to the analytic situation in which the inviscid weak solution requires boundary data only on the 'inflow' part of the boundary at which characteristics enter the domain.



**Figure 5: case (i)**

$$(ii) \quad a_+ < 0, a_- > 0, b_{\pm} > 0 \quad (|b| \gg |a|)$$

Equation (3.4a) reduces to

$$a_- u_{i-1/2, j} + (a_- - a_+) u_{i+1/2, j} - a_+ u_{i+3/2, j}$$

$$-2a_- u_{i, j-1/2} + 2a_+ u_{i+1, j-1/2} \simeq 0.$$

Again  $u_{i+1/2, j}$  is influenced only by upwind points as shown below.

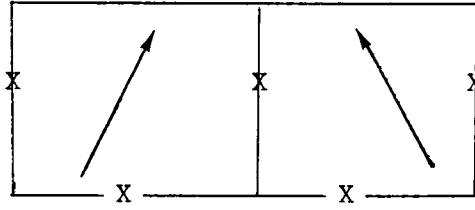


Figure 6: case (ii)

$$(iii) \quad a_+ > 0, a_- < 0, b_{\pm} > 0 \quad (|b| \gg |a|)$$

Here

$$u_{i+1/2, j} - \frac{1}{2} (u_{i, j-1/2} + u_{i+1, j-1/2}) \simeq 0.$$

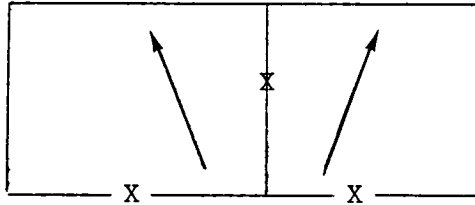


Figure 7: case (iii)

In interpreting these observations it must be remembered that corresponding equations arise from (3.4b). These examples provide useful insights in understanding the test problems considered in the next section. In these problems  $\rho$  varies from 0.3 to 40 with corresponding values of  $\theta$  ranging from 0 to 40 so the actual finite difference equations can be very close to the different limiting forms in different parts of the domain.

## 5. NUMERICAL EXAMPLES

We have indicated earlier that we may expect that the compact scheme (2.2) yields a solution  $u(v, h)$  which converges to the solution  $u(v)$  of (2.1) with an  $L_2$  error of order  $h^2$  (c.f. Example 1, below). In this section we are primarily interested in studying the  $L_2$  error  $E$  between  $u(v, h)$  and the inviscid solution  $u^0$  as  $h \rightarrow 0$ ,  $v \rightarrow 0$ , by means of numerical experiments.

The first question which arises concerns the possibility of solving (3.4) for  $u(v, h)$  when  $\rho \rightarrow \infty$ . Although several iterative methods were studied with, generally, successful results our report will concentrate on the use of the Gauss-Seidel method. One reason for our doing so stems from favorable results, as yet unpublished, by our colleague T. N. Phillips concerning the treatment of systems closely related to (3.4) by multigrid methods using Gauss-Seidel as the underlying iterative scheme.

A central question motivating this study is how effectively internal and external boundary layers arising from approximating  $u^0$  by  $u(v, h)$  can be localized. A related question concerns the relative influence of the viscosity  $\nu$  compared to inaccuracies arising from the fact that the scheme has truncation errors of order  $h^2$ . In order to help discuss this we

introduce a parameter  $\tau = h^2/\nu$  which together with  $\rho = h/\nu$ , will be used to study the solution as  $h \rightarrow 0$ ,  $\nu(h) \rightarrow 0$ .

Another issue concerns new insights into upwind difference schemes. As discussed earlier  $|\theta| \ll 1$  leads to a diffusion-type limit of (3.4) while  $|\theta| \gg 1$  leads to an inviscid upwind-type treatment. Both can occur if  $\rho \gg 1$  and  $a$  (or  $b$ ) passes through zero in some region. The functions  $p, q$  in (3.4) automatically handle this transition with a "viscous" treatment near  $a \sim 0$  and an "almost-inviscid" treatment elsewhere.

The final matter concerns the question: under what limiting processes do nonconservative finite difference schemes yield the correct asymptotic inviscid result when  $h \rightarrow 0$ ,  $\nu(h) \rightarrow 0$ ?

The examples which follow address these questions. The first two examples graphically illustrate the behaviour of  $u(\nu, h)$  in cases in which the inviscid limit  $u^0$  is either linear or nonlinear. In the next five examples the  $L_2$  error  $E$  between the solution  $u(\nu, h)$  and the inviscid, analytic solution  $u^0$  is computed for different values of  $\nu, h$  and the results are compared using simple heuristic analyses. The final example calculates the  $L_2$  error for a problem using a nonconservative form of (3.4).

#### Example 1

$f$  and  $g$  are linear and are defined by

$$\begin{aligned} f &= a u & a &= x - 1/2 \\ g &= b u & b &= 1 - y \end{aligned}$$

Note that  $f_x + g_y = a u_x + b u_y$  since  $a_x + b_y = 0$ . Hence the inviscid solution  $u^0$  is constant along characteristics defined parametrically by

$$\frac{dx}{ds} = a, \quad \frac{dy}{ds} = b, \quad \text{i.e., } (1 - y)(1/2 - x) = \text{constant.}$$

The domain considered is  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$  and so the only inflow boundary on which boundary data needs to be specified is on  $y = 0$ . The boundary values are chosen to be

$$u^0(x, 0) = \sin \pi(1/2 - x),$$

so that the solution on the interior is

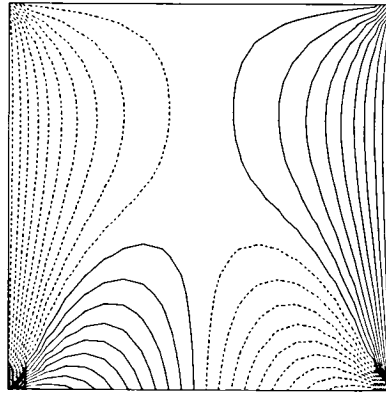
$$u^0(x, y) = \sin[\pi(1 - y)(1/2 - x)].$$

The boundary conditions for the finite difference problem are chosen to be

$$u(x, 1) = u^0(x, 1), \quad u(0, y) = -1, \quad u(1, y) = 1$$

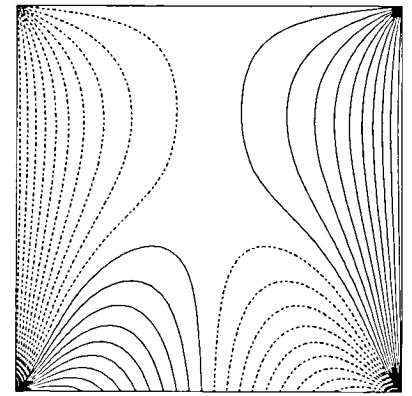
so that boundary layers arise at  $x = 0, 1$ .

Figure 8 shows contour plots of  $u(v, h)$  for several different values of  $v$  and  $h$ . (a), (b), (c) illustrate ordinary convergence arising from a sequence in which  $v$  is kept fixed and  $h$  is reduced by factor  $1/2$  each time; little visual difference appears. (a), (d), (e) are a sequence in which  $\rho = h/v$  is kept fixed and  $h$  is reduced by factor 2 each time; observe that the width of the boundary layers decreases. Finally, (a), (f), (g) illustrate, a sequence in which  $h$  is kept fixed and  $v$  is decreased by a factor 4 each time. Note that the boundary layers at  $x = 0, 1$  become smaller until they are confined to one interior cell.



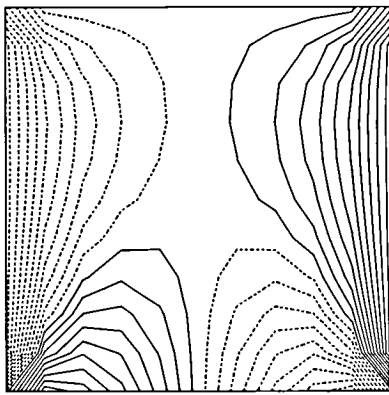
(b)

$$\nu = 1/5, N = 20$$



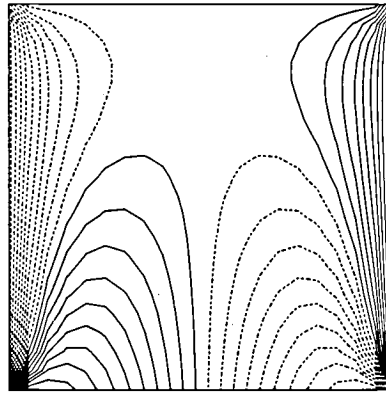
(c)

$$\nu = 1/5, N = 40$$



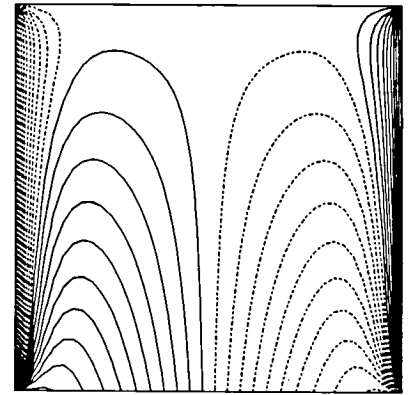
(a)

$$\nu = 1/5, N = 10$$



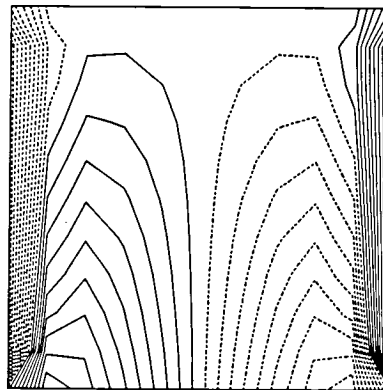
(d)

$$\nu = 1/10, N = 20$$



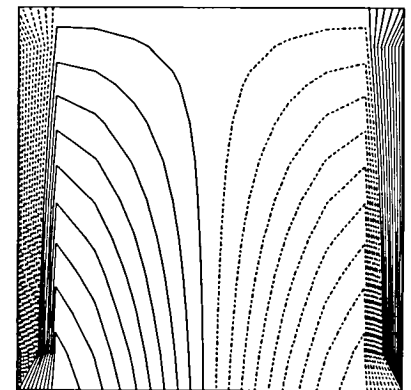
(e)

$$\nu = 1/20, N = 40$$



(f)

$$\nu = 1/20, N = 10$$



(g)

$$\nu = 1/80, N = 10$$

**Figure 8.** Contour plots for Example 1; (a), (b), (c) illustrate ordinary convergence ( $h \rightarrow 0$ ,  $\nu = \text{const.}$ ); in (a), (d), (e)  $\rho = \text{const.}$ ,  $h \rightarrow 0$ ; in (a), (f), (g),  $h = \text{const.}$ ,  $\nu \rightarrow 0$ . (In (f), (g) boundary layer thickness is confined to one cell.)

Example 2

This nonlinear example was suggested by our colleague E. Tadmor to test the ability of the scheme to converge to the correct "physical" inviscid solution for a problem with an infinite range of formal inviscid solutions. We consider

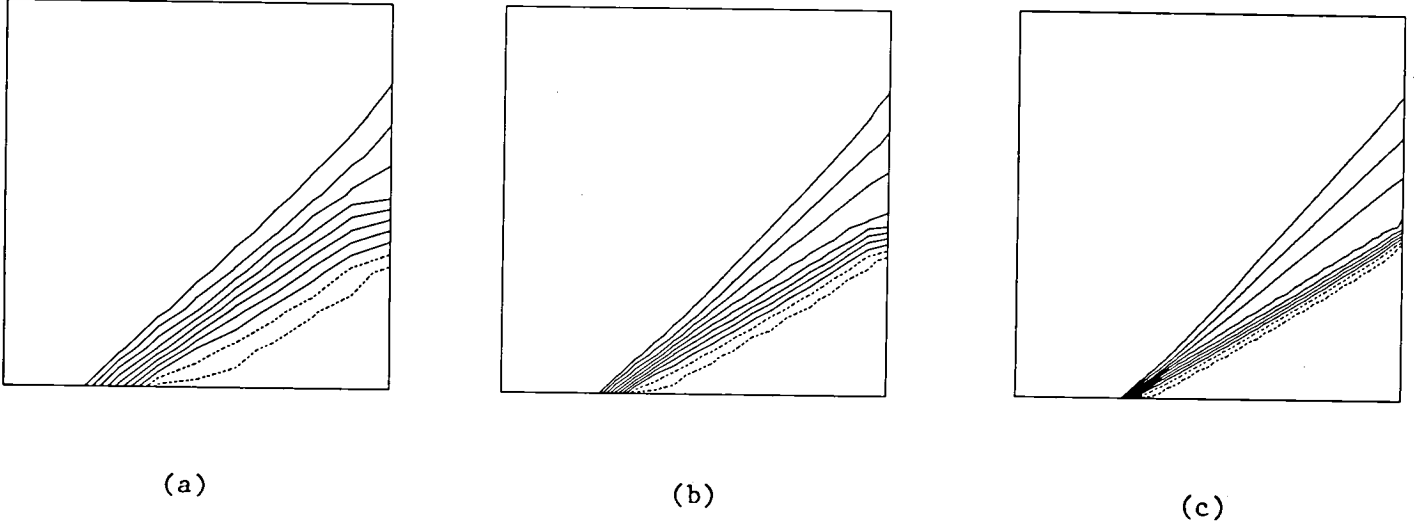
$$f = u^2, \quad g = u^3.$$

The test case has an expansion fan attached to a shock

$$u^0(x,y) = \begin{cases} 0.8 & \frac{x - 0.3}{y} < \frac{1}{1.2} \\ \frac{2y}{3(x - 0.3)} & \frac{1}{1.2} < \frac{x - 0.3}{y} < \frac{1}{0.6} \\ -0.2 & \frac{1}{0.6} < \frac{x - 0.3}{y} \end{cases}.$$

The boundary conditions for the finite difference solution are  $u(v,h) = u^0$  on all four sides so that the only sharp gradients occur at the shock.

Figure 9 shows contour plots of  $u(v,h)$  for fixed  $\rho$  and decreasing  $h$ . In (a) the shock is indistinguishable from the expansion fan but in (c) the difference is obvious, and it is also clear that the numerical solution is converging to the physical inviscid solution.



**Figure 9.** Example 2,  $f = u^2$ ,  $g = u^3$ ;  $\rho = \text{const.}$ ,  $h \rightarrow 0$ .

### Example 3

This linear example uses the same definition of  $f$  and  $g$  as Example 1, and has the same inviscid solution  $u^0$ . However this time the finite difference scheme has boundary conditions  $u = u^0$  on all four sides of the square domain. Three cases were run, one with a  $20 \times 20$  grid, and two with  $80 \times 80$  grids keeping  $\rho = h/\nu$  fixed in one case, and  $\tau = h^2/\nu$  fixed in the other. The table below shows the (i) number of Gauss-Seidel iterations, (starting from initial conditions  $u = 0$ ); (ii) the  $L_2$  error  $E$ ; (iii)  $E$  divided by the error  $E^*$  for the  $20 \times 20$  grid, and (iv) the predicted ratio  $E/E^*$  based on the following simple analysis.

Let  $L_0$  be the inviscid differential operator and  $L(\nu, h)$  be the viscous finite difference operator. Then

$$L_0 u^0 = 0 \quad \text{and} \quad L(\nu, h) u(\nu, h) = 0,$$

so

$$L(\nu, h)(u^0 - u(\nu, h)) = (L(\nu, h) - L_0)u^0 \sim h^2 \cdot \text{function} \left( \frac{h^2}{\nu} \right)$$



and  $u^0 - u(v, h) = 0$  on boundary; thus

$$\|u^0 - u(v, h)\|_{L_2} \sim h^2 e(\tau) \quad \text{as } h \rightarrow 0,$$

where  $e(\tau)$  is some function of the truncation error due to viscosity.

Table I. Results for Example 3

	N = 20		N = 80	
	$\rho = 10$		$\rho = 40$	
	$\tau = 0.5$		$\tau = 0.125$	
No. of Iterations	15		55	25
E	.334 (-2)		.821 (-3)	.216 (-3)
E/E*	1		0.245	0.0647
Predicted Ratio	1		0.25	0.0625

#### Example 4

This example has linear  $f, g$

$$f = 0.5 u, \quad g = u$$

with discontinuous boundary data for which the inviscid solution  $u^0$  is

$$u^0 = \begin{cases} 0.8 & x < 0.3 + 0.5y \\ -0.4 & x > 0.3 + 0.5y \end{cases}.$$

The analysis for this case is not easy but leads to the result

$$E \sim h^{1/2} e(\tau),$$

where  $e$  is a function with asymptotic behaviour,

$$\tau \rightarrow 0 \quad e \sim \tau^{-1/4} \implies E \sim v^{1/4}.$$

Thus when  $\tau$  is small the error  $E$  is due not to the second order truncation but due rather to the viscous smearing of the contact discontinuity.

Table II. Results for Example 4

	N = 20		N = 80	
	$\rho = 10$		$\rho = 10$	$\rho = 40$
	$\tau = 0.5$		$\tau = 0.125$	$\tau = 0.5$
No. of Iterations	10		15	15
E	.144		.0969	.0901
E/E*	1		0.673	0.625
Predicted Ratio	1		0.707	0.5

#### Example 5

This example is a two-dimensional version of Berger's equation with a shock:

$$f = 0.5 u^2, \quad g = 0.4 u$$

and the inviscid solution  $u^0$  is

$$u^0 = \begin{cases} 0.8 & x < 0.3 + 0.5y \\ -0.4 & x > 0.3 + 0.5y \end{cases}.$$

In the finite difference solution the shock becomes an internal boundary layer with width of order  $\nu$ . If  $\rho = h/\nu$  is kept fixed as  $h \rightarrow 0$ , this shock layer is spread over the same number of cells and hence

$$E \sim h^{1/2} e(\rho).$$

As  $\rho \rightarrow 0$ ,  $e \sim \rho^{-1/2} \implies E \sim \nu^{1/2}$  and so when  $\rho$  is small the error is due solely to a physically well-resolved shock.

Table III. Results for Example 5

	N = 20		N = 80	
	$\rho = 10$		$\rho = 10$	$\rho = 40$
	$\tau = 0.5$		$\tau = 0.125$	$\tau = 0.5$
No. of Iterations	35		85	85
E	.102		.050	.048
E/E*	1		.49	.47
Predicted Ratio	1		.5	.5

#### Example 6

This example is exactly the same as Example 1 except for different values of  $h$  and  $\nu$ . The analysis cannot be expressed in one simple equation. If  $\rho \rightarrow 0$  and  $h \rightarrow 0$  then  $E \sim \nu^{1/2}$  since the boundary layer has thickness of order  $\nu$  and so it is well-resolved. If  $\rho$  is kept fixed the boundary layer

spans a fixed number of cells and so  $E \sim h^{1/2}$ . If  $\tau$  is kept fixed it has an  $O(h)$  effect on just the interior points adjacent to the boundary and so  $E \sim h^{3/2}$ .

Table IV. Results for Example 6

	N = 20	N = 80	
	$\rho = 10$	$\rho = 10$	$\rho = 40$
	$\tau = 0.5$	$\tau = 0.125$	$\tau = 0.5$
No. of Iterations	15	55	25
E	.448 (-1)	.254 (-1)	.51 (-2)
E/E*	1	0.567	0.114
Predicted Ratio	1	0.5	0.125

#### Example 7

This example is the same of Example 3 except that the boundary condition at  $y = 1$  is  $u(x,1) = 1$  which produces a boundary layer at  $y = 1$ . This boundary layer is different from the ones in Example 6. In Example 6 the boundaries were 'outflow' boundaries, i.e., the inviscid characteristics were pointing outwards across the boundary. In this example the characteristics are tangential to the boundary at  $y = 1$ , similar to a stagnation point flow in fluid dynamics.

The physical boundary has thickness of order  $v^{1/2}$  so if  $\tau$  is kept fixed as  $h \rightarrow 0$  it spans a fixed number of cells and  $E \sim h^{1/2}$ . Hence

$$E \sim h^{1/2} e(\tau)$$

and as  $\tau \rightarrow 0$ ,  $e \sim \tau^{-1/4} \implies E \sim v^{1/4}$ .

**Table V. Results for Example 7**

	N = 20		N = 80	
	$\rho = 10$		$\rho = 40$	
	$\tau = 0.5$		$\tau = 0.125$	
No. of Iterations	20	60	25	
E	.149	.107	.074	
E/E*	1	0.718	0.498	
Predicted Ratio	1	0.707	0.5	

Table VI summarizes the results of Examples 3-7.

**Table VI. Summary of Asymptotic Behaviour of E**

E	$v = \text{const}$	$\rho = \text{const}$	$\tau = \text{const}$
Smooth $u^0$	$v$	$h$	$h^2$
Linear, discontinuous $u^0$	$v^{1/4}$	$h^{1/4}$	$h^{1/2}$
Nonlinear, discontinuous $u^0$	$v^{1/2}$	$h^{1/2}$	$h^{1/2}$
Smooth $u^0$ , "wrong" outflow boundary conditions	$v^{1/2}$	$h^{1/2}$	$h^{3/2}$
Smooth $u^0$ , "wrong" tangent boundary conditions	$v^{1/4}$	$h^{1/4}$	$h^{1/2}$

Example 8

This example studies a nonconservative form of the compact equations obtained by replacing the term  $\delta_x f + \delta_y g$  in (2.2a) by  $\bar{a} \delta_x u + \bar{b} \delta_y u$  where  $\bar{a}, \bar{b}$  are average values of  $f_u$  and  $g_u$  in the cell. The test case is

$$f = u^4, \quad g = 0.4 u$$

with an inviscid solution

$$u^0 = \begin{cases} 0.8 & x < 0.3 + 0.5y \\ -0.4 & x > 0.3 + 0.5y \end{cases}.$$

The error analysis shows that  $E$  has two components,  $E_1$  due to shock smearing, and  $E_2$  due to an incorrect shock angle.

$$E_1 \sim (\text{shock width})^{1/2}$$

$$\sim v^{1/2} = h^{1/2} \rho^{-1/2}$$

$$E_2 \sim (\text{conservation loss per unit shock length})^{1/2}$$

$$\sim [(\# \text{ cells across shock}) \times (\# \text{ cells along shock}) \times (\text{truncation error per cell})]^{1/2}$$

$$\sim \left[ \frac{v}{h} \times \frac{1}{h} \times \frac{h^3}{v^2} \right]^{1/2}$$

$$\sim \rho^{1/2}.$$

The two error components have the same order of magnitude when  $\rho \sim h^{1/2}$ .  
 If  $\rho \ll h^{1/2}$ ,  $E_1$  dominates and, if  $\rho \gg h^{1/2}$ ,  $E_2$  dominates. Hence,

$$E \sim \begin{matrix} v^{1/2} & \rho \ll h^{1/2} \\ \rho^{1/2} & \rho \gg h^{1/2} \end{matrix}$$

and at each fixed  $h$ ,  $E$  is minimized by choosing  $\rho \sim h^{1/2}$ , and  
 $E_{\min} \sim h^{1/4}$ .

Table VIIa shows results from several experiments and Table VIIb has the corresponding values of  $E_{\min}/h^{1/4}$  and  $\rho_{\min}/h^{1/2}$  which, according to the analysis, should be constants.

Table VIIa:

E	N	
	$\rho$	
	20	80
0.31	.267	.162
0.62	.220	.121
1.25	.168	.133
2.5	.228	.266
5	.376	.393

Table VIIb

	N	
	20	80
$E_{\min}/h^{1/4}$	.355	.362
$\rho_{\min}/h^{1/2}$	.18	.18

## 6. CONCLUDING REMARKS

The experiments reported in the previous section suggest that the compact scheme (2.2) provides an effective means of approximating both  $u(v)$  and  $u^0$  and that the relationship between these solutions which are suggested by singular perturbation arguments is maintained, as indicated by Figure 1, by the finite difference scheme as well. A fact of potential practical importance is that boundary layers can be confined to a single computational cell. This feature is a relevant factor as well in selecting the conservative form of (2.2) instead of the nonconservative form (we remark in passing that it would have been preferable to have treated the terms  $p\delta$ ,  $q\delta$  in (2.2) by  $\delta p$ ,  $\delta q$  instead).

Finally, the fact that the flux-eliminated equations (3.4) can be treated by a Gauss-Seidel iterative method indicates that a wide variety of more rapid iterative methods can also be employed. In contrast, the use of time-stepping methods to obtain  $u(v)$  or  $u^0$  can be seen to result in a more slowly convergent Jacobi-type iteration scheme.



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